Collinearity/Concurrence*

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1 Introduction/Facts you should know

1. (Cevian Triangle) Let ABC be a triangle and P be a point. Let lines AP, BP, CP meet lines BC, AC, AB at D, E, F, respectively. Triangle DEF is called a *cevian triangle of P with respect to ABC*.

2. (Ceva) Let D, E, and F be points on sides BC, AC and AB, respectively, of triangle ABC. Then AD, BE, and CF are concurrent if and only if $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$.

3. (Trig Ceva) In triangle ABC, let D, E, and F be points on sides BC, AC, and AB respectively. Then AD, BE and CF are concurrent if and only if
$$\frac{\sin\angle BAD}{\sin\angle ABE} \cdot \frac{\sin\angle CBE}{\sin\angle BCF} \cdot \frac{\sin\angle ACF}{\sin\angle CAD} = 1.$$

4. (Ceva in a circle) Let A, B, C, D, E, F be six consecutive points on a circle. We have AD, BE, CF are concurrent if and only if $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$.

5. (Menelaus) Let ABC be a triangle and let D, E, and F be points on lines BC, AC, and AB, respectively. Then D, E, and F are collinear if and only if $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$.

6. (Isogonal conjugates) Let ABC be a triangle and P be a point not equal to any of A,B, C. The reflections of lines AP, BP, CP over the angle bisectors of A, B, C, respectively, concur at a point. This point is called the *isogonal conjugate of ABC*.

7. (Harmonic conjugates) Let ABC be a triangle and DEF be a cevian triangle. Let $EF \cap AB = P$. Then P, D are harmonic with respect to B, C. That is $\frac{PB}{PC} \cdot \frac{DC}{DB} = -1$.

8. (Desargues) Let $A_1B_1C_1$, $A_2B_2C_2$ be triangles in space. Lines A_1A_2, B_1B_2, C_1C_2 are concurrent (or all parallel) if and only if the intersections of corresponding sides $A_1A_2 \cap B_1B_2$, $A_2A_3 \cap B_2B_3$, and $A_3A_1 \cap B_3B_1$ are collinear.

*Some material and problems taken from MOP 2012 collinearity and concurrency handout by Carlos Shine and http://www.math.cmu.edu/~ploh/docs/math/6-concur-solns.pdf

9. (Pappus) Let A_1, A_2, A_3 and B_1, B_2, B_3 be two sets of collinear points. Then $A_1B_2 \cap A_2B_1, A_1B_3 \cap A_3B_1, A_2B_3 \cap A_3B_2$ are collinear.

10. (Pascal) Let $ABCDEF$ be six points on a conic. Then the intersections of $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear. The hexagon does not need to be convex, and degenerate cases are allowed. For example, if we took the hexagon $AABCDE$, then AA is the tangent through A.

11. (Radical center) Let $\omega_1, \omega_2, \omega_3$ be circles. Then the radical axes of ω_1, ω_2, ω_2, ω_3, and ω_3, ω_1 are either all parallel or concurrent at the radical center of the three circles.

12. (Brianchon) Let $ABCDEF$ be a hexagon circumscribed to a circle. Then AD, BE, CF are concurrent. The hexagon does not need to be convex, and degenerate cases allowed.

2 Warmups

1. Prove the existence of Isogonal conjugates.

2. (Gergonne Point) Let ABC be a triangle. Let A_1, B_1, C_1 be the points where the incircle touches sides BC, AC and AB, respectively. Prove AA_1, BB_1, CC_1 are concurrent.

3. Let ABC be a triangle with $AB = AC$. Let ℓ be a line that intersects sides BC, AC, and AB at D, E, and F respectively. Suppose that $DE = DF$. Show that $CE = BF$.

4. Let ABC be a triangle. A line through M, the midpoint of BC, parallel to the angle bisector of $\angle A$ meets sides AC and AB at E and F, respectively. Show that $CE = BF$.

5. $ABCD$ trapezoid with $AB||CD$. AC and BD meet at P and AD and BC meet at Q. Show that PQ passes through the midpoints of AB and CD.

6. (Euler line) Prove the circumcenter, orthocenter, and centroid of a triangle are collinear.

3 Easier Problems

7. (Cevian Nest Theorem) Let $\triangle Q_1Q_2Q_3$ be the cevian triangle of a point P with respect to $\triangle P_1P_2P_3$, and let $\triangle R_1R_2R_3$ be the cevian triangle of a point Q with respect to $\triangle Q_1Q_2Q_3$. Prove that lines P_1R_1, P_2R_2, P_3R_3 are concurrent.

8. (IMO 1961) Let D, E, and F be points on sides BC, AC and AB, respectively, of triangle ABC, such that AD, BE, and CF are concurrent at point P. Show that among the numbers $\frac{AP}{PD}, \frac{BP}{PE}, \frac{CP}{PF}$, at least one is ≥ 2 and at least one is ≤ 2.

9. (Own) In cyclic quadrilateral $ABCD$, $AB * BC = AD * DC$. Let X, Y be on CD, BC respectively such that $BX||AD, DY||AB$. Prove $XY||BD$.

10. (USAMO 2003) Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

11. (IMO Shortlist 2001) Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent.

12. Let ABC be a triangle and A_1, B_1, C_1 be the points where the incircle touches sides BC, AC, and AB, respectively. Prove that if A_2, B_2, C_2 are points on minor arcs B_1C_1, C_1A_1, A_1B_1, respectively, of the incircle of ABC such that AA_2, BB_2, CC_2 are concurrent, then A_1A_2, B_1B_2, C_1C_2 are concurrent.

13. (Russia 1997) Given triangle ABC, let A_1, B_1, C_1 be the midpoints of the broken lines CAB, ABC, BCA, respectively. Let ℓ_A, ℓ_B, ℓ_C be the respective lines through A_1, B_1, C_1 parallel to the angle bisectors of A, B, C. Show that ℓ_A, ℓ_B, ℓ_C are concurrent.

14. (IMO 1982) Let $ABCDEF$ be a regular hexagon, with points M and N on diagonals AC and CE respectively, such that $\frac{AM}{AC} = \frac{CN}{CE} = r$. Find r, if B, M, N are collinear.

15. (Seven Circles Theorem) Let $C, C_1, C_2, C_3, C_4, C_5, C_6$ be circles such that C is externally tangent to C_i at P_i for all i, and C_i is externally tangent to C_{i+1} for all i, with C_6 externally tangent to C_1. Prove that P_1P_4, P_2P_5, P_3P_6 are concurrent.

16. (Classic) Let ABC be a triangle with incenter I. Let Γ be the circle tangent to sides AB, AC, and the circumcircle of ABC. Let Γ touch sides AB and AC at X and Y, respectively. Prove I is the midpoint of XY.

17. (Composition of homotheties) If σ_1, σ_2 are homotheties with centers O_1, O_2 respectively and ratios k_1, k_2 (possibly negative), and $k_1k_2 \neq 1$, then the composition $\sigma_1 \circ \sigma_2$ is a homothety with center O and ratio k_1k_2 and O, O_1, O_2 are collinear. Try proving this with Desargues' theorem.

18. (Monge's Theorem) Let $\omega_1, \omega_2, \omega_3$ be three circles such that no circle contains another circle. Let P_1 be the intersection points of the common external tangents of ω_2 and ω_3, and define P_2, P_3 similarly. Show that P_1, P_2, and P_3 are collinear.

4 Problems

19. (MOP 1998) Let ABC be a triangle, and let A' , B' , C' be the midpoints of the arcs BC, CA, AB, respectively, of the circumcircle of ABC. The line $A'B'$ meets BC and AC at S and T. $B'C'$ meets AC and AB at F and P, and $C'A'$ meets AB and BC at Q and R. Prove that the segments PS, QT, FR concur.

20. (Bulgaria 1997) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that H, O, D are collinear.

21. (IMO Shortlist 1997) The bisectors of angles A, B, C of triangle ABC meet its circumcircle again at the points K, L, M, respectively. Let R be an internal point on side AB. The points P and Q are defined by the conditions: RP is parallel to AK and BP is perpendicular to BL; RQ is parallel to BL and AQ is perpendicular to AK. Show that the lines KP, LQ, MR concur.

22. Let ABC be a triangle, D, E, F be the points of tangency of the incircle with BC, CA, and AB, respectively, and A', B', C' be the midpoints of arcs BC, AC, and AB, respectively, on the circumcircle. If I is the incenter, and M, N, P are the midpoints of the segments ID, IE, IF, show that lines MA', NB', PC' pass through the same point.

23. (IMO Shortlist 1997) Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices mod 3). Let $B_i, i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

24. (IMO Shortlist 2006) Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

25. (TSTST 2017) Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

26. Three equal circles $\omega_1, \omega_2, \omega_3$ are tangent to the pairs of sides of triangle ABC that meet at A, B, and C, respectively, and are also internally tangent to the circle Ω. Prove that the center of Ω lies on the line that passes through the incenter and the circumcenter of ABC.

27. Let ω be a circle and $ABCD$ be a square somewhere in its interior. Construct the circles $\omega_a, \omega_b, \omega_c, \omega_d$ to be exterior to the square but internally tangent to the circle ω, at some points denoted by A', B', C', D', respectively, and such that they are tangent to the pairs of lines AB and AD, AB and BC, BC and DC, and CD and DA, respectively. Prove that the lines AA', BB', CC', DD' meet at one point.

28. (IMO Shortlist 2007) Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent

to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

29. (IMO 2008) Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by ω_1 and ω_2 the incircles of triangles ABC and ADC. Suppose that there exists a circle ω inscribed in angle ABC, tangent to the extensions of line segments AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.